# Kinetic Models Beyond Classical Binary Collisions

## Mass Exchange, Bose-Einstein Condensation, and Wave Kinetic

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P. Degond and J.-G. Liu, *Binary particle collisions with mass exchange*, JSP (2025),

S. Luo and J.-G. Liu, *On the spatially homogeneous Boltzmann equation with mass exchange*, arXiv:2607.20684.

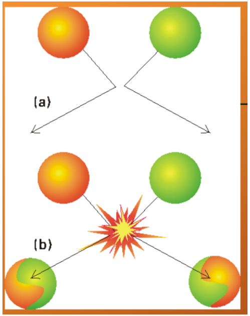
S. Luo and J.-G. Liu, *No gelation and global existence for a Boltzmann equation with regularly varying mass-exchange rates*, arXiv:2607.25112.

# Binary particle collisions with mass exchange

$$\boxed{(m, \boldsymbol{v}) + (m_1, \boldsymbol{v}_1) \quad \xrightarrow{\text{collision}} \quad (m', \boldsymbol{v}') + (m_1', \boldsymbol{v}_1')}$$

- **Mass conservation**: $m + m_1 = m' + m_1'$
- **Elastic collision** (momentum & energy):

$$m\boldsymbol{v} + m_1\boldsymbol{v}_1 = m'\boldsymbol{v}' + m_1'\boldsymbol{v}_1'$$
$$m|\boldsymbol{v}|^2 + m_1|\boldsymbol{v}_1|^2 = m'|\boldsymbol{v}'|^2 + m_1'|\boldsymbol{v}_1'|^2$$




- Mass exchange rate: $A_{m,m_1;m'}$ (symmetric in $m, m_1$)
- Velocity change rate (cross section): $\mathbf{B}\left(\frac{mm_1}{m+m_1}|\boldsymbol{v}-\boldsymbol{v}_1|^2, \omega \cdot \frac{\boldsymbol{v}-\boldsymbol{v}_1}{|\boldsymbol{v}-\boldsymbol{v}_1|}\right)$,

$$\boldsymbol{v}' = \boldsymbol{v}_{\mathrm{CM}} + \alpha\sqrt{\frac{m_1}{m}}(I - 2\omega\otimes\omega)(\boldsymbol{v}-\boldsymbol{v}_1), \qquad \boldsymbol{v}_{\mathrm{CM}} = \frac{m\boldsymbol{v} + m_1\boldsymbol{v}_1}{m+m_1}$$

$$\boldsymbol{v}_1' = \boldsymbol{v}_{\mathrm{CM}} - \alpha\sqrt{\frac{m}{m_1}}(I - 2\omega\otimes\omega)(\boldsymbol{v}-\boldsymbol{v}_1), \qquad \alpha = \frac{\sqrt{mm_1}}{m+m_1}, \quad \omega\text{(deflection angle)}$$

# Boltzmann Equation with Mass Exchange: An Infinite System

$$\boxed{\frac{\partial}{\partial t} f_m(\boldsymbol{x},\boldsymbol{v},t) + \boldsymbol{v}\cdot\nabla_x f_m(\boldsymbol{x},\boldsymbol{v},t) = Q_m(f), \quad \forall m \in \mathbb{N}, \quad \forall \boldsymbol{x},\boldsymbol{v} \in \mathbb{R}^{\mathrm{d}},\ t > 0}$$

## Collision Operator

$$Q_m(f)(\boldsymbol{v}) = \int\limits_{\substack{(v_1,\Omega)\in\mathbb{R}^{\mathrm{d}}\times\mathbb{S}^{\mathrm{d}-1} \\ (\boldsymbol{v}-\boldsymbol{v}_1)\cdot\Omega\le 0}} \mathbf{B}\left(\frac{mm_1}{m+m_1}|\boldsymbol{v}-\boldsymbol{v}_1|^2, \Omega\cdot\frac{\boldsymbol{v}-\boldsymbol{v}_1}{|\boldsymbol{v}-\boldsymbol{v}_1|}\right)$$

$$\times\left[\sum_{\substack{(m',m_1') \\ m'+m_1'\ge m+1}} A_{m',m_1';m} f_{m'} f_{m_1'} \left(\frac{mm_1}{m'm_1'}\right)^{\frac{\mathrm{d}}{2}} - \sum_{m_1=1}^{\infty}\sum_{m'=1}^{m+m_1-1} A_{m,m_1;m'} f_m f_{m_1}\right] \mathrm{d}\boldsymbol{v}_1 \mathrm{d}\Omega$$

# Detailed balanced size distribution for multiplicative exchange rates

Multiplicative Rate Assumption:

There exists $\{\gamma_m\}_{m\geq 1} \subset [0,\infty)$ such that:

$$\boxed{A_{m,m_1;m'} = \gamma_m \gamma_{m_1} \quad \text{for all } m, m_1 \geq 1 \text{ and } m' \in \{1,\dots,m+m_1-1\}}$$

For pure mass exchange ($\boldsymbol{v}, \boldsymbol{v}_1 = 0$):

$$\boxed{m + m_1 \quad \underset{\gamma_{m'}\gamma_{m_1'}}{\overset{\gamma_m \gamma_{m_1}}{\rightleftharpoons}} \quad m' + m_1'}$$

Normalized detailed balanced size distribution: for some $\beta < \beta_0$

$$f_m^\beta := \frac{e^{\beta m}}{\gamma_m Z^\beta}, \quad Z^\beta := \sum_{m\geq 1} \frac{m e^{\beta m}}{\gamma_m}, \qquad \sum_{m\geq 1} m f_m^\beta = 1$$

Expectation w.r.t. equilibrium size distribution:

$$\langle a_m \rangle_\beta = \sum_{m\geq 1} a_m \, m f_m^\beta$$

# Entropy dissipation with multiplicative rates

Entropy Dissipation Inequality:

$$\sum_{m\geq 1}\int_{\mathbb{R}^{\mathrm{d}}} Q_m(f)\log\left(\frac{\gamma_m f_m}{m^{\mathrm{d}/2}}\right)\mathrm{d}\boldsymbol{v} \leq 0$$

**H-Theorem - Entropy Decay:**
Define the entropy (convex) functional:

$$H[f] = \sum_{m\geq 1}\int_{\mathbb{R}^{\mathrm{d}}}\left[f_m\log f_m + \left(\log\gamma_m - \tfrac{\mathrm{d}}{2}\log m\right) f_m\right]\mathrm{d}\boldsymbol{v}$$

which satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t}H[f] \leq 0$$

# Hydrodynamics limit

$$\boxed{\partial_t f_m^\varepsilon + \boldsymbol{v} \cdot \nabla_x f_m^\varepsilon = \frac{1}{\varepsilon} Q_m(f^\varepsilon), \quad \forall m \geq 1, \quad \forall \boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^{\mathrm{d}}, \, t > 0}$$

As Knudsen number $\varepsilon \to 0$,

$$f^\varepsilon \to f = \rho M_{\boldsymbol{u},\Theta,\beta}$$

and $\mathcal{M} = (\mathcal{N}, \rho, \boldsymbol{p}, E)$ satisfy the Euler system of $\mathrm{d} + 3$ equations

$$\begin{aligned} \partial_t \mathcal{N} + \nabla \cdot (\mathcal{N} \boldsymbol{u}) &= 0 \\ \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) &= 0 \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p &= 0, \qquad E = \tfrac{\rho}{2} |\boldsymbol{u}|^2 + \frac{p}{\gamma - 1}, \\ \partial_t E + \nabla \cdot ((E + p) \boldsymbol{u}) &= 0 \end{aligned}$$

where $\gamma = 1 + 2/\mathrm{d}$. It is the monatomic gas.

# Navier-Stokes Mass-Exchange (NS-ME) Model as BGK-ME Approximation

## BGK-ME Kinetic Equation

$$\partial_t f_m^\varepsilon + \boldsymbol{v} \cdot \nabla_x f_m^\varepsilon = \frac{1}{\varepsilon}\Big(\rho_{f^\varepsilon}\big(M_{\boldsymbol{u}_{f^\varepsilon},\Theta_{f^\varepsilon},\beta_{f^\varepsilon}}\big)_m - f_m^\varepsilon\Big), \quad \forall m \geq 1$$

## NSME using the Chapman-Enskog procedure

$$\begin{aligned}
\partial_t \mathcal{N} + \nabla \cdot (\mathcal{N}\boldsymbol{u}) &= \nabla \cdot (\nu \nabla_x \chi), \\
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) &= 0, \\
\partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p &= \nabla \cdot (\mu \sigma(\boldsymbol{u})), \\
\partial_t E + \nabla \cdot ((E+p)\boldsymbol{u}) &= \nabla \cdot \Big((\tfrac{\mathrm{d}}{2}+1)\nu \Theta \nabla_x \chi + \kappa \nabla \Theta + \mu \sigma(\boldsymbol{u})\boldsymbol{u}\Big),
\end{aligned}$$

# Positivity of Transport Coefficients

## Viscosity and Conductivity Coefficients

The Navier-Stokes mass-exchange system features three strictly positive transport coefficients:

$$\mu = \varepsilon\rho\Theta\langle m^{-1}\rangle_\beta > 0 \quad \text{(Shear viscosity)}$$

$$\kappa = \varepsilon\frac{\mathrm{d}+2}{2}\rho\Theta\langle m^{-2}\rangle_\beta > 0 \quad \text{(Thermal conductivity)}$$

$$\nu = \varepsilon\rho\Theta\left(\langle m^{-2}\rangle_\beta - \langle m^{-1}\rangle_\beta^2\right) > 0 \quad \text{(Mass diffusion coefficient)}$$

## Auxiliary Operators

- Stress tensor:

$$\sigma(\boldsymbol{u}) = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \frac{2}{\mathrm{d}}(\nabla\cdot\boldsymbol{u})I_{\mathrm{d}}$$

- Thermodynamic potential:

$$\chi = \log\left(\frac{\rho\Theta}{Z^\beta}\right)$$

# Entropy Dissipation in NSME System

## Theorem (Entropy Inequality)

*For smooth solutions* $(\rho, \boldsymbol{u}, \Theta, \beta)$ *of NSME, the entropy* $S$ *satisfies:*

$$\partial_t S + \nabla \cdot \tilde{\boldsymbol{\phi}} = -D \leq 0$$

*where the dissipation* $D$ *is:*

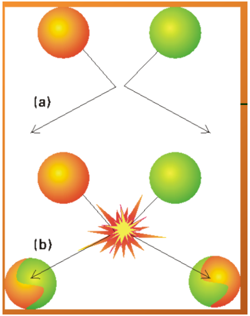
$$D = \underbrace{\nu |\nabla \chi|^2}_{\textit{Mass diffusion}} + \underbrace{\kappa \left| \frac{\nabla \Theta}{\Theta} \right|^2}_{\textit{Heat conduction}} + \underbrace{\frac{\mu}{2\Theta} \sigma(\boldsymbol{u}) : \sigma(\boldsymbol{u})}_{\textit{Viscous dissipation}} \geq 0$$

The NSME entropy flux $\tilde{\boldsymbol{\phi}}$ modifies the equilibrium flux $\boldsymbol{\phi} = S\boldsymbol{u}$ by:

$$\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi} + \varepsilon \left[ -\nu \left( \log \rho - \log Z - 1 - \tfrac{\mathrm{d}}{2} \right) \nabla \chi + \kappa \frac{\nabla \Theta}{\Theta} \right]$$

# Binary particle collisions with continuous mass exchange, notations

$$\boxed{(m, \boldsymbol{v}) + (m_1, \boldsymbol{v}_1) \quad \xrightarrow{\text{collision}} \quad (m', \boldsymbol{v}') + (m'_1, \boldsymbol{v}'_1)}$$




$$\alpha := \frac{m'}{S}, \quad \theta := \frac{m}{S}, \quad S := m + m_1$$

$$s = \sqrt{\frac{\theta(1-\theta)}{\alpha(1-\alpha)}}, \quad \mathcal{R}_\omega = I - 2\omega \otimes \omega$$

$$v_{\mathrm{CM}} = \frac{m\boldsymbol{v} + m_1\boldsymbol{v}_1}{m + m_1}$$

$$\boldsymbol{v}' = \boldsymbol{v}_{\mathrm{CM}} + (1-\alpha)s\mathcal{R}_\omega(\boldsymbol{v} - \boldsymbol{v}_1),$$

$$\boldsymbol{v}'_1 = \boldsymbol{v}_{\mathrm{CM}} - \alpha s\mathcal{R}_\omega(\boldsymbol{v} - \boldsymbol{v}_1),$$

- Mass exchange rate: $a(m, m_1, \alpha)$ symmetric in $m, m_1$
- Velocity change rate (cross section): $\mathbf{B}\left(\frac{mm_1}{m+m_1}|\boldsymbol{v} - \boldsymbol{v}_1|^2, \omega \cdot \frac{\boldsymbol{v}-\boldsymbol{v}_1}{|\boldsymbol{v}-\boldsymbol{v}_1|}\right) = \mathbf{B}\left(E_{\mathrm{red}}, \omega \cdot \frac{\boldsymbol{u}}{|\boldsymbol{u}|}\right)$

# Spatially Homogenous Continuous-mass BME

Take $X = (0,\infty)_m \times \mathbb{R}^{\mathrm{d}}_{\boldsymbol{v}}$. The spatially homogeneous Boltzmann equation with mass exchange reads

$$\boxed{\partial_t f(t,\boldsymbol{x}) = \mathbf{Q}(f(t),f(t))(\boldsymbol{x}), \qquad f(0,\boldsymbol{x}) = f_0(\boldsymbol{x}) \geq 0, \qquad t>0, \boldsymbol{x}\in X.} \tag{BME}$$

The weak formulation of the collision operator is given by

$$\begin{aligned}
\mathcal{Q}(f,f)[\psi] &:= \frac{1}{2}\int_{X^2}\int_{\mathbb{S}^{\mathrm{d}-1}}\int_0^1 a(m,m_1,\alpha)\mathbf{B}(E_{\mathrm{red}},\omega\cdot\hat{\boldsymbol{u}})f(\boldsymbol{x})f(\boldsymbol{x}_1)\Delta\psi \mathrm{d}\alpha\mathrm{d}\omega\mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x},\\
\Delta\psi &:= \psi(\boldsymbol{x}') + \psi(\boldsymbol{x}_1') - \psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_1), \qquad \text{for all } \psi\in B_b(X),\\
\mathcal{Q}(f,f)[\psi] &= \int_X \mathbf{Q}(f,f)(\boldsymbol{x})\psi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \qquad \text{for all } \psi\in B_b(X).
\end{aligned}$$

# Kernel Assumptions

**Mass exchange rate.**

(K1B) $a : (0,\infty)^2 \times (0,1) \to [0,\infty)$ is measurable, **bounded**, and continuous.

(K1L) $a : (0,\infty)^2 \times (0,1) \to [0,\infty)$ is measurable, continuous and satisfies

$$a(m, m_1, \alpha) \lesssim 1 + m + m_1.$$

(K3) $a(m, m_1, \alpha) = a(m_1, m, \alpha) = a(m, m_1, 1-\alpha)$ for all $m, m_1 > 0$ and $\alpha \in (0,1)$.

(RV) There exist a PDF $g \in C([0,1]; [0,\infty))$, $g(\alpha) = g(1-\alpha)$, a function $\lambda \in C((0,\infty)^2; (0,\infty))$ such that $\lambda(m, m_1) = \lambda(m_1, m)$, and a regular varying $\kappa \in C((0,\infty); (0,\infty))$ with index $p \in [0,1]$
such that there exists $\lambda_-, \lambda_+, \kappa_+ > 0$, $\kappa(S) \le \kappa_+(1+S)$ and

$$\boxed{a(m, m_1, \alpha) = \lambda(m, m_1) g(\alpha), \quad \lambda_- \kappa(S) \le \lambda(m, m_1) \le \lambda_+ \kappa(S).}$$

$f$ Regularly varying with index $p$: if $\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^p$ for all $t > 0$.

# Kernel Assumptions

**Hard-potential collision rate.**

(K2) $\mathbf{B} : (0,\infty) \times \mathbb{S}^{d-1} \to [0,\infty)$ is measurable, continuous and satisfies, for $\gamma \in (0,1)$,

$$\boxed{\mathbf{B}(E_{\mathrm{red}},\xi) = E_{\mathrm{red}}^{\gamma} b(\xi), \qquad \sup_{e\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} b(e\cdot\omega)\,\mathrm{d}\omega =: \|b\|_{L^1(\mathbb{S}^{d-1})} < \infty.}$$

**Initial value.**

(IC1) $f_0 \in L^1(X;(1+m+m|\boldsymbol{v}|^2)\mathrm{d}\boldsymbol{x})$ and $f_0 \geq 0$ a.e. in $X$.

(IC2) $f_0 \in L^1(X;(m|\boldsymbol{v}|^2)^{1+\delta}\mathrm{d}\boldsymbol{x})$ for some $\delta > 0$.

(IC3) $f_0 \in L^1(X;(m|\boldsymbol{v}|^2)^{1+\gamma}\mathrm{d}\boldsymbol{x})$.

(Hp) $f_0 \in L^1(X;(1+m+m|\boldsymbol{v}|^2)^{p}\mathrm{d}\boldsymbol{x})$ for some $p \geq 1+\gamma$.

# Moment

For a nonnegative density $h$ on $X$, set

$$M_0(h) := \int_X h(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \qquad M_1(h) := \int_X m h(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$
$$M_2(h) := \int_X m|\boldsymbol{v}|^2 h(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{P}(h) := \int_X m\boldsymbol{v} h(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$
$$H_p(h) := \int_X (1 + m + m|\boldsymbol{v}|^2)^p h(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \quad p \geq 1.$$

For a finite nonnegative Radon measure $\mu$ on $X$, we define the moments of $\mu$ by replacing $h(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ with $\mathrm{d}\mu(\boldsymbol{x})$ in the above definitions.

# Bounded mass-exchange kernels

**Theorem (Global $L^1$-theory, Luo–Liu, 2026)**

*Assume bdd mass-exchange rate, hard-potential $\gamma \in (0,1)$, finite initial physical moments. Then* (BME) *admits a global nonnegative $L^1$-integral weak solution $f$ satisfying*

$$f \in W^{1,\infty}(0,\infty; L^1(X)) \cap L^\infty(0,\infty; L^1(X; (1+m+m|\boldsymbol{v}|^2)\,\mathrm{d}\boldsymbol{x})),$$

$$\mathbf{Q}(f,f) \in L^\infty(0,\infty; L^1(X)), \qquad \partial_t f = \mathbf{Q}(f,f) \quad \text{in } L^1(X) \text{ a.e.}$$

*For every $t \geq 0$,*

$$\boxed{M_0(f(t)) = M_0(f_0), \qquad M_1(f(t)) = M_1(f_0), \qquad M_2(f(t)) \leq M_2(f_0).}$$

*If $1+\delta, \delta > 0$-order moment holds, then $M_2(f(t)) = M_2(f_0)$ for $t \geq 0$.*
*If $1+\gamma$-order moment holds, $f$ is the unique global solution in the energy-dissipating class.*

# Local theory for linearly growing mass-exchange kernels

### Theorem (Local $H_p$-theory, Luo–Liu, 2026)

*Assume linearly growing/regular varying mass-exchange kernels, hard-potential $\gamma \in (0,1)$, finite initial physical moments, and $1+\gamma$-order moment. Set $\vartheta = \gamma/(p-1)$, and*

$$\Lambda_p = C_{p,\gamma} A_a \|b\|_{L^1} H_1(f_0)^{1-\vartheta}, \qquad T_p = \frac{H_p(f_0)^{-\vartheta}}{2\vartheta\Lambda_p}.$$

*Then there exists a nonnegative conservative $L^1$-integral weak solution on $[0, T_p]$ such that*

$$f \in L^\infty(0, T_p; L^1(X; (1+m+m|\boldsymbol{v}|^2)^p \, \mathrm{d}\boldsymbol{x})) \cap W^{1,\infty}(0, T_p; L^1(X)),$$

### Theorem (Critical-moment continuation criterion, Luo–Liu, 2026)

*Let $f$ be a conservative $H_p$-solution on $[0,T)$, where $T<\infty$. If $H_{1+\gamma}(f) \in L^1(0,T)$, then $f$ extends to a conservative $H_p$-solution on $[0, T+\varepsilon)$ for some $\varepsilon > 0$.*

# Global theory for regularly varying mass-exchange kernels

## Theorem (Global $L^1$-existence, Luo–Liu, 2026)

*Assume linearly growing/regular varying mass-exchange kernels, hard-potential $\gamma \in (0,1)$, finite initial physical moments. Then there exists a global nonnegative $L^1$-integral weak solution $f$ satisfying*

$$f \in W^{1,\infty}(0,\infty; L^1(X)) \cap L^\infty(0,\infty; L^1(X; (1+m+m|\boldsymbol{v}|^2)\,\mathrm{d}\boldsymbol{x})$$

## Theorem (No mass gelation, Luo–Liu, 2026)

*For every $T > 0$, let $(f_N)_{N\geq 1}$ be the mass-cutoff approximation sequence of $f$ on $[0,T]$. Then*

$$\boxed{\lim_{R\to\infty} \sup_{N\geq 1} \sup_{0\leq t\leq T} \int_{\{m>R\}} m f_N(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \lim_{R\to\infty} \sup_{0\leq t\leq T} \int_{\{m>R\}} m f(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad = 0.}$$

*Thus no mass escapes to $m = \infty$ on any finite time interval.*

# Dust Condensation at $m = 0$

Take $a_N B_N = \min\{aB, N\}$ and consider the cutoff approximation solutions with initial data $f_0$. A compactness argument of $(f_N)$ may yield a weak solution $f$ to (BME) in the (weak) limit.

## Missing compactness near $m = 0$ and possible number condensation

- The physical bound does not imply

$$\lim_{r \downarrow 0} \sup_{N \geq 1} \sup_{0 \leq t \leq T} \int_0^r \int_{\mathbb{R}^d} f_N(t, m, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}m = 0.$$

- Since $m|\boldsymbol{v}|^2$ does not control large velocities when $m \ll 1$, we cannot give tail estimates for large $\boldsymbol{v}$.
- Consequently, it is possible for a weak measure-valued limit to develop a zero-mass dust condensation

$$\mu_t(\mathrm{d}m, \mathrm{d}\boldsymbol{v}) = f_t(m, \boldsymbol{v}) \mathrm{d}m \mathrm{d}\boldsymbol{v} + \delta_0(\mathrm{d}m) \otimes \nu_t(\mathrm{d}\boldsymbol{v}).$$

  at finite time for $t > T_{\mathrm{con}}$. After the condensation time, the dust may also interact with regular particles.

# Mass gelation at $m = \infty$

Now let's consider the case when the mass-kernel is linearly growing. For every cutoff approximation, $M_1$-moment is conserved. However, this does not give the required mass tightness (no-gelation):

$$\boxed{\lim_{R\to\infty} \sup_{N\geq 1} \sup_{0\leq t\leq T} \int_{\{m>R\}} m f_N(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0.}$$

## A local $H_p$ Estimate

- For the local case, a standard substitute is to assume a higher weighted moment: $H_p(f_0) < \infty, p \geq 1+\gamma$. Hence

$$H_p(f_N(t)) \leq \left(H_p(f_0)^{-\vartheta} - \vartheta\Lambda_p t\right)^{-1/\vartheta}, \qquad \vartheta = \gamma/(p-1), \quad 0 \leq t < T_p.$$

- But for $T \geq T_p$, we cannot get the mass-tightness estimate, and the mass may escape to $m = \infty$ in finite time. As a result, for a finite time $T_{\text{gel}}$, $M_1(f_t) < M_1(f_0)$ for $t > T_{\text{gel}}$. **Gelation might happen in some cases.**

# No-Condensation: small-mass bootstrap

For $r > 0$, write $F_n(r,t) := \int_0^r \int_{\mathbb{R}^d} f_n(t,m,\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}m$. Test the cutoff equation with $\mathbf{1}_{\{m<r\}}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}F_n(r,t) \leq \int_{X^2}\int_0^1\int_{\mathbb{S}^{d-1}} a_n E^\gamma b f_n(\boldsymbol{x}) f_n(\boldsymbol{x}_1)(\mathbf{1}_{\{\alpha(m+m_1)<r\}}+\mathbf{1}_{\{(1-\alpha)(m+m_1)<r\}})\mathrm{d}\omega\mathrm{d}\alpha\mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x}.$$

Devide $X^2 = \{m+m_1<\rho\} \cup \{m+m_1 \geq \rho\}$ for some $1 > \rho > r > 0$.

- On $\{m+m_1<\rho\}$, $\int_{\{m+m_1<\rho\}} a_n E^\gamma f_n(\boldsymbol{x}) f_n(\boldsymbol{x}_1)\mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x} \leq C F_n(\rho,t)^{2-\gamma}$.
- On $\{m+m_1 \geq \rho\}$,
  $\int_0^1 a_n(m,m_1,\alpha)(\mathbf{1}_{\{\alpha(m+m_1)<r\}} + \mathbf{1}_{\{(1-\alpha)(m+m_1)<r\}})\mathrm{d}\alpha \leq 2A_a r(1+\rho^{-1})$.

Hence

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}F_n(r,t) \leq C\|b\|_1 F_n(\rho,t)^{2-\gamma} + 4A_a\|b\|_1 M_0(f_0)^{2-\gamma}M_2(f_0)^{\gamma} r(1+\rho^{-1}).}$$

Taking $\rho = \sqrt{r}$ and integrating in $[t_0, t_0+\tau]$
Define

$$L_I = \lim_{r\downarrow 0} \sup_N \sup_{t\in I} F_N(r,t).$$

If $\lim_{r\downarrow 0} \sup_N F_N(r,t_0) = 0$, then taking the upper limit yields

$$L_I \leq C\tau L_I^{2-\gamma}.$$

Because $0 \leq L_I \leq M_0(f_0)$, choose $\tau > 0$, independently of $t_0, N$, so that $C\tau M_0(f_0)^{1-\gamma} < 1$.
Then implies

$$L_I \leq C\tau M_0(f_0)^{1-\gamma} L_I < L_I$$

unless $L_I = 0$; hence $L_I = 0$.

# Uniform integrability, estimates on good and bad collision sets

Define

$$U_N(q,t) := \sup_{\substack{A\subset X \text{ measurable} \\ |A|\leq q}} \int_A f_N(t,x)\,\mathrm{d}x.$$

Define

$$\Lambda_I = \limsup_{q\downarrow 0} \sup_N \sup_{t\in I} U_N(q,t),$$

with some careful estimates on good and bad collision sets, we obtain

$$\Lambda_I \leq 2\tau R_{\varepsilon,\kappa,L} + 2C_1\tau\mathcal{B}(\sigma).$$

where the angular absolute-continuity modulus

$$\mathcal{B}(\sigma) = \sup_{e\in\mathbb{S}^{d-1}} \sup_{\substack{O\subset\mathbb{S}^{d-1} \text{ measurable} \\ |O|\leq\sigma}} \int_O b(e\cdot\omega)\,\mathrm{d}\omega.$$

One has $\mathcal{B}(\sigma)\to 0$ as $\sigma\downarrow 0$.

# No Gelation: hinge-token mass-moment $\Phi$

To get the no-gelation, one need to analyze the **incomparable collisions**.

## A hinge-token for comparable and incomparable collisions

Take $H_L(m) = (m-L)_+$. Consider the case where $m = m_1 = 50$ and $L = 50$.

- For **comparable collisions**, $\alpha = 0.45$, we have $H_L(m') + H_L(m_1') = H_L(45) + H_L(55) = 5$.
- For **incomparable collisions**, $\alpha = 0.05$, we have $H_L(m') + H_L(m_1') = H_L(5) + H_L(95) = 45$.

Hence the hinge-token can identify whether the collision output is comparable or not, and we can use it to control the mass escaping to $m = \infty$.

Exactly, we choose sequence of levels $L_k = 2^k L_0$ and choose weights $q_k$ and construct $\Phi(m)$ such that

$$\boxed{\Phi(m) = m + \sum_{k=0}^{\infty} q_k H_{L_k}(m), \quad \int_X \Phi(m) f_0(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} < \infty, \quad \lim_{m\to\infty} \frac{\Phi(m)}{m} = \infty.}$$

# Construction of weights $q_k$

Choose $L_0 \geq 1$ so large that

$$\mathcal{T}_0 := \int_{\{m > L_0\}} m f_0(x)\,\mathrm{d}x \leq \frac{1}{2},$$

and use dyadic levels

$$L_k = 2^k L_0, \qquad \mathcal{T}_k = \int_{\{m > L_k\}} m f_0(x)\,\mathrm{d}x, \qquad k \geq 0.$$

$$2^{-1/2} q_k \leq q_{k+1} \leq q_k, \qquad \sum_{k=0}^{\infty} q_k = \infty, \qquad \sum_{k=0}^{\infty} q_k \mathcal{T}_k < \infty.$$

# Localized and regularized collision forms, convergence

Choose

$$K_q = [r_q, R_q] \times \overline{B_{R_q}(\mathbf{0})} \Subset X, \qquad \sup_{N, t\in[0,T]} \int_{K_q^c} f_N(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \le q, \qquad \sup_{t\in[0,T]} \int_{K_q^c} f(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \le q.$$

Here $x = (m, v)$, $x_1 = (m_1, v_1)$, $\widehat{\boldsymbol{u}} = (\boldsymbol{v} - \boldsymbol{v}_1)/|\boldsymbol{v} - \boldsymbol{v}_1|$, and $E = \dfrac{mm_1}{m+m_1}|\boldsymbol{v} - \boldsymbol{v}_1|^2$.

**1. Localization of the incoming variables**

$$\mathcal{Q}^{K_q}(g,g)[\psi] := \frac{1}{2}\iint_{K_q^2}\int_0^1\int_{\mathbb{S}^{d-1}} a(m,m_1,\alpha)E^\gamma b(\widehat{\boldsymbol{u}}\cdot\omega)\,\Delta\psi\, g(\boldsymbol{x})g(\boldsymbol{x}_1)\,\mathrm{d}\omega\,\mathrm{d}\alpha\,\mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x}.$$

**2. Exclusion of the mass-exchange endpoints**

$$\mathcal{Q}^{K_q,\rho}(g,g)[\psi] := \frac{1}{2}\iint_{K_q^2}\int_\rho^{1-\rho}\int_{\mathbb{S}^{d-1}} a(m,m_1,\alpha)E^\gamma b(\widehat{\boldsymbol{u}}\cdot\omega)\,\Delta\psi\, g(\boldsymbol{x})g(\boldsymbol{x}_1)\,\mathrm{d}\omega\,\mathrm{d}\alpha\,\mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x}.$$

**3. Approximation of the angular kernel**

$$\mathcal{Q}^{K_q,\rho,j}(g,g)[\psi] := \frac{1}{2}\iint_{K_q^2}\int_\rho^{1-\rho}\int_{\mathbb{S}^{d-1}} a(m,m_1,\alpha)E^\gamma b_j(\widehat{\boldsymbol{u}}\cdot\omega)\,\Delta\psi\, g(\boldsymbol{x})g(\boldsymbol{x}_1)\,\mathrm{d}\omega\,\mathrm{d}\alpha\,\mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x}.$$

The continuous approximation $b_j \in C([-1,1])$ is chosen so that

$$\delta_j := \sup_{e\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \Big|b_j(e\cdot\omega) - b(e\cdot\omega)\Big|\,\mathrm{d}\omega \longrightarrow 0.$$

# Summary of the mass-exchange work

## Main results

- Built an $L^1$-**integral weak solution theory** for the continuous-mass BME.
  - Global theory for **bounded** mass-exchange kernels.
  - Local $H_p$-theory for **linearly growing** kernels.
  - **Global no-gelation theory** for regularly varying kernels.

## Main mechanisms

- Replaced entropy compactness by a direct **compactness bootstrap**:
  - **no-dust small-mass estimate** near $m = 0$;
  - collision-geometry **uniform integrability** via the area formula.
- Replaced high-moment control by a tailor-made superlinear weight $\Phi$.
  - hinge-token detects **incomparable mass-exchange**;
  - **four-region decomposition** absorbs the main positive flux.

## Conceptual point

physical moments $+$ collision geometry $\Longrightarrow$ global compactness **without entropy assumptions.**

# Possible application 1: isotropic Bose–Boltzmann equation

- Nordheim (1928) introduced a quantum version of the Boltzmann collision law.
- Uehling and Uhlenbeck (1933) developed the corresponding kinetic equation. The spatially homogenous Bose-Boltzmann equation reads

$$\partial_t f(t,\boldsymbol{v}) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \boldsymbol{B}(\boldsymbol{v}-\boldsymbol{v}_1,\omega)[f' f_1' (1+f)(1+f_1) - f f_1 (1+f')(1+f_1')]\mathrm{d}\omega\mathrm{d}\boldsymbol{v}.$$

- The basic collision geometry remains the same as the classical Boltzmann equation, but the occupation factor $1+f$ records the bosonic transition rate. The factor reduces to $(1+f+f_1)$, compare with $f+f_1$ in wave kinetic equation.
- If the interaction potential $\phi(\cdot)$ is of the central form, i.e. $\phi(\boldsymbol{x}) = \phi(|\boldsymbol{x}|)$, in the weak-coupling regime, according to Erdös–Salmhofer–Yau (2004), Benedetto–Pulvirenti–Castella–Esposito (2005) and Cai–Lu (2019, 2026), the collision kernel

$$\boldsymbol{B}(\boldsymbol{v}-\boldsymbol{v}_1,\omega) = \frac{1}{(4\pi)^2}|(\boldsymbol{v}-\boldsymbol{v}_1)\cdot\omega|\Phi(|\boldsymbol{v}-\boldsymbol{v}'|,|\boldsymbol{v}-\boldsymbol{v}_1'|),$$

$$\Phi(r,\rho) = (\hat{\phi}(r)+\hat{\phi}(\rho))^2, \hat{\phi}(r) = \mathcal{F}\phi(\boldsymbol{\xi})\mid_{|\boldsymbol{\xi}|=r}.$$

# Bose–Einstein Condensation

Physically, Bose–Einstein condensation means that a macroscopic fraction of the particles occupies the ground state $\boldsymbol{v} = \mathbf{0}$. For an isotropic distribution, set $x = \frac{1}{2}|\boldsymbol{v}|^2$ and $F_t(\mathrm{d}x)$ be the measure-valued solution. Then $F_t$ has a BEC at time $t$ iff. $F_t(\{0\}) > 0$.
Take

$$N_t := \int_{[0,\infty)} \mathrm{d}F_t(x), \quad E_t := \int_{[0,\infty)} x\mathrm{d}F_t(x), \quad \frac{T}{T_c} = \left[\frac{E_0}{\Gamma(\frac{5}{2})\zeta(\frac{5}{2})}\right]^{2/5} \left[\frac{\Gamma(\frac{3}{2})\zeta(\frac{3}{2})}{N_0}\right]^{2/3}.$$

The Bose–Einstein equilibrium where $T/T_c < 1$[1] is

$$F_{\mathrm{be}}(\mathrm{d}x) = \frac{\sqrt{x}\mathrm{d}x}{\exp(x/T) - 1} + N_0\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]\delta_0(\mathrm{d}x).$$

**Question:** If $F_0(\{0\}) = 0$, is it possible that $F_{t_*}(\{0\}) > 0$ for some $t_* < \infty$? Besides, is it true that $F_t(\mathrm{d}x)$ converges to $F_{\mathrm{be}}(\mathrm{d}x)$ and $F_t(\{0\})$ converges to $N_0\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] =: N_c$?

[1] This is equivalent to that the BEC happens in the equilibrium.

## Cai–Lu 2026 prior work for $\Phi(r,\rho) \approx (r^2+\rho^2)^\eta$ as $r^2+\rho^2 \downarrow 0$

Cai–Lu assume $0 \le \Phi \in C_b(\mathbb{R}_+^2), \Phi(r,\rho) = \Phi(\rho,r)$.

- To obtain the condensation result, they assume $\rho \mapsto \Phi(r,\rho)$ is nondecresing and $b_0 \min\{1,(r^2+\rho^2)^\eta\} \le \Phi(r,\rho) \le 1$.
- To obtain the noncondensation result, they assume $\Phi \in C^2$ and $0 \le \Phi \le \min\{1,(r^2+\rho^2)^\eta\}$.

They obtain

- For $\eta \in [0,1/4)$, $F_t(\{0\}) \to N_c, \|F_t - F_{\rm be}\|_1 \to 0, F_{t_*}(\{0\}) > 0$ for finite $t_*$.
- For $\eta \in [1/4,1)$, choose $\alpha \in (0,1-\eta)$, **if additionally** $E_0/N_0 \le [4(3/2+\alpha)]^{-1}$ and $F_0([0,\varepsilon]) \ge \varepsilon^\alpha$ for $\varepsilon \ll 1$, then $F_t(\{0\}) \to N_c, \|F_t - F_{\rm be}\|_1 \to 0, F_{t_*}(\{0\}) > 0$ for finite $t_*$.n
- For $\eta \ge 1$, $F_t(\{0\}) \le e^{48\sqrt{2}N(0)^2 t} F_0(\{0\})$.

**Problem:** In the assumption for $\eta \in [1/4,1)$, one can imply $T/T_c \ll 1$, which restricts the result in **ultra-low-temperature regime**. Besides, it also assumes a **quantitative concentration of the initial mass** near $x=0$, leaving untreated initial data whose mass is initially located away from the origin.

**Question:** Can we remove the additional ultra-low-temperature and near-zero-condensation assumptions?

# Possible result we are working on

## Possible result

Assume $1/4 \le \eta < 1, T < T_c$ and $b_0 \min\{1, (r^2 + \rho^2)^\eta\} \le \Phi(r, \rho) \le 1$. For the conservative solution constructed by Cai–Lu, there exists $t_* < \infty$ such that

$$F_{t_*}(\{0\}) > 0, \qquad F_t(\{0\}) \to F_{\mathrm{be}}(\{0\}).$$

- This will remove the Cai–Lu additional assumptions

$$T/T_c \ll 1, \qquad F_0([0, \varepsilon]) \ge C\varepsilon^\alpha.$$

- What we are currently working: a revised version of small-mass bootstrap and hinge-token moments from the mass-exchange model can provide a proof for the result.

# Possible application 2: the 4-wave kinetic equation

Peierls (1928) first introduced the wave kinetic equation (also called the photon kinetic equation). Note that occupation factor $n_1 + n_2$ instead of $1 + n_1 + n_2$ in BEC, due to massless nature of photons and no conservation of particle number.
For dispersion relation $\omega(\boldsymbol{k}) = |\boldsymbol{k}|^\alpha$, the **3D 4-wave kinetic equation**, formally derived by Benney–Saffman (1966) and rigorously derived in $\alpha = 2$ from cubic NLS by Deng–Hani (2023) reads

$$\partial_t n(t, \boldsymbol{k}_1) = \mathcal{C}[n](\boldsymbol{k}_1),$$

$$\mathcal{C}[n](\boldsymbol{k}_1) = \int_{\mathbb{R}^9} \boldsymbol{\delta}(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)[n_3 n_4 (n_1 + n_2) - n_1 n_2 (n_3 + n_4)] \, \mathrm{d}\boldsymbol{k}_{234}.$$

**Staffilani–Tran:** $1 < \alpha \leq 2$

- **Finite-time action condensation:** under suitable radial initial-data conditions, the solution $n(t, \{0\}) > 0$ for some $t < \infty$.
- **Energy cascade:** the energy leaves every bounded frequency region as $t \to \infty$.

**Future possible applications to the mass-exchange model mechanism:**

- **(small-mass bootstrap)** Finite-time **action condensation** phase transition for $\alpha > 0$.
- **(hinge-token moment)** Finite-time **energy cascade** phase transition for $\alpha > 0$.

# The quasilinear kernel $|\omega_1\omega_2\omega_3\omega_4|^\beta$

Fix the Schödinger dispersion $\omega(\boldsymbol{k}) = |\boldsymbol{k}|^2$, and consider the quasilinear NLS

$$i\partial_t u + \Delta u = |\nabla|^\beta(|\nabla|^\beta u||\nabla|^\beta u|^2), \qquad |\nabla| = \sqrt{-\Delta}, \qquad \beta \in [0,1]$$

The corresponding quasilinear 4-wave equation introduced by Ampatzoglou–Léger (2024) is

$$\partial_t n(t,\boldsymbol{k}_1) = \mathcal{C}[n](\boldsymbol{k}_1),$$
$$\mathcal{C}[n](\boldsymbol{k}_1) = |\boldsymbol{k}_1|^{2\beta}\int_{\mathbb{R}^9}(|k_2||k_3||k_4|)^{2\beta}\boldsymbol{\delta}(\boldsymbol{\Sigma})\delta(\Omega)[n_3n_4(n_1+n_2) - n_1n_2(n_3+n_4)]\,\mathrm{d}\boldsymbol{k}_2\mathrm{d}\boldsymbol{k}_3\mathrm{d}\boldsymbol{k}_4,$$

where $\boldsymbol{\Sigma} := \boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4, \Omega := \omega_1 + \omega_2 - \omega_3 - \omega_4$.

**Possible works:**

- Finite-time action condensation possible transition at $\beta = 1/4$.
- Finite-time energy cascade possible transition at $\beta = 2/3$. **(a mass exchange model inspired threshold)**

# Future works for BEC and wave kinetic equation

**Bose–Einstein condensation:**

- Condensation for inverse-power and non-cutoff collision kernels
- Dynamics after condensation
- Non-radial and space-dependent condensation

**Wave kinetic equation:**

- Unconditional energy escape example for $\alpha > 3$
- action condensation example for $\alpha < 1$
- sharp condensation & cascade 2D transition diagram for $(\alpha, \beta)$.

**Thank you!**